J. Fluid Mech. (1986), *vol.* 168, *pp.* 73–87
Printed in Great Britain

The dynamics of magnetic fields in a highly conducting turbulent medium and the generalized Kolmogorov–Fokker–Planck equations

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(Received 9 July 1984 and in revised form 21 October 1985)

It is shown that a consideration of the magnetic field in a highly conducting turbulent medium, using Lagrange variables, involves deriving kinetic equations of fluid-particle transition probability densities. A derivation of such equations is performed for joint probability densities of n particles up to $n = 4$. By assuming normality of one particle distribution function it was found that these kinetic equations are the generalized Kolmogorov–Fokker–Planck (KFP) equations. The dynamics of mean and fluctuating magnetic fields is described by means of these equations. The eddy diffusivity of a mean field for processes described by generalized KFP equations coincides with that of a scalar field (depending in general on helicity in implicit form). It is shown that at sufficiently large magnetic Reynolds number, a turbulence with any spectrum generates fluctuating magnetic fields.

1. Introduction

Kinetic equations are widely used to describe dynamic systems with fluctuating parameters. In this paper such equations will be employed in order to clarify some properties of the magnetic field in a highly conducting turbulent fluid.

There are well-known kinetic equations, governing the random Markov processes with no probability consequences and these are the Kolmogorov–Fokker–Planck (KFP) equations of the diffusion type. As regards turbulence, the Markov-type of process implies a statistical independence of motions of the medium at different, though close, moments of time. This is equivalent to assuming a δ-correlation in time of the velocity of random motions. Physically this implies the use of a small parameter, $\tau u/l \ll 1$ (τ, u and l are the correlation time, the characteristic velocity and the scale of energy-containing eddies respectively). For real turbulence, typically $\tau \approx l/u$, and the above-indicated small parameter is absent. Therefore, for describing the dynamics of magnetic fields in turbulent media it is desirable to employ kinetic equations that are valid for random non-Markovian processes.

Derivation of such equations is of wider interest than the application to magnetic fields. A number of authors (Bartlett 1955; Roberts 1961; Kraichnan 1966; Pawula 1967; Rytov 1976), on examining non-Markovian processes, have concluded that continuous processes such as these obey second-order differential equations that coincide in structure with KFP equations but have, generally, other coefficients. Such equations are referred to as generalized KFP equations, and have been treated most thoroughly by Pawula (1967).

The KFP equations contain sufficient information for describing the dynamics of magnetic fields. The problem of the magnetic field in a turbulent fluid at large magnetic Reynolds numbers is of particular interest in numerous astrophysical applications. For a medium of low conductivity, the amplitudes of magnetic-field fluctuations are small and perturbation theory is invoked to treat the mean fields (Steenbeck, Krause & Rädler 1966; Steenbeck & Krause 1966; Moffatt 1978; Krause & Rädler 1980). For high conductivity this is not the case but it is possible to overcome the difficulties that arise by using a model of random motion that is δ-correlated in time (the Markov process).

For large-scale fields (LSF) the use of such a model (Vainshtein 1972), by itself, appears not to limit the validity range of the results obtained. The form of an equation for the mean field

$$\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) = -\nabla \times (D_{\mathrm{T}} \nabla \times \boldsymbol{B}) + \nabla \times \alpha \boldsymbol{B}$$

can be determined from dimensionality considerations. The problem is merely that of accurately calculating the coefficients D_{T} and α. However, for an LSF dynamo to occur, an accurate evaluation of these quantities is not decisive. For any D_{T} and α, fields of a sufficiently large scale L, satisfying the inequality $L \gg D_{\mathrm{T}}/\alpha$, will be generated by a turbulent fluid. The only problem concerns the sign of the magnetic-field eddy diffusivity, whose solution requires an accurate theory. As we will see in §4, D_{T} is positive and coincides with the eddy diffusivity of a scalar field.

A quite different situation is typical of the fields of scales l', $l' \lesssim l$, which will henceforth be referred to as small-scale fields (SSF). Hence, the Markovian model (Kazantsev 1967), though allowing us to clarify certain regularities, does not provide any solution to the problem. At large magnetic Reynolds numbers and not too small scales l' (i.e. large compared with the dissipative scale l_0, for which the 'local' magnetic Reynolds number is of order unity), microscopic magnetic diffusion is not significant. The dynamics of fields of such scales is determined by two competing processes: field amplification by turbulent motions as a result of an extension of the lines of force; and magnetic-energy transfer across the spectrum into the region of small scales (eddy magnetic diffusion). The rates of these processes are order-of-magnitude coincident. Therefore, simple estimations do not provide any solution to the problem of SSF dynamo. An accurate theory is necessary, capable of treating real turbulence ($\tau \sim l/u$), as was outlined by Kraichnan & Nagarajan (1967).

The possibility of constructing an accurate theory is opened up by the Lagrange approach, widely used at present (Moffatt 1974; Kraichnan 1976*a*, *b*, 1979; Moffatt 1978, 1983; Vainshtein 1982). In the next section we will describe a version of the Lagrange approach that, when applied to magnetic fields, employs kinetic equations for the probability densities of fluid-particle transitions. Within the framework of such an approach we will show in §5 that the SSF dynamics is dominated by the field amplification and an SSF dynamo takes place.

2. Lagrangian description of the magnetic-field dynamics

A magnetic field $\boldsymbol{H}$ in a highly conducting medium, that moves with a velocity $\boldsymbol{v}$, satisfies the induction equation

$$\frac{\partial \boldsymbol{H}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{H}). \tag{1}$$

An exact solution in Lagrangian coordinates is known to be

$$H_i[\boldsymbol{x}(\boldsymbol{a},t),t] = \frac{\rho[\boldsymbol{x}(\boldsymbol{a},t)]}{\rho(\boldsymbol{a},0)} H_j(\boldsymbol{a},0) \frac{\partial x_i(\boldsymbol{a},t)}{\partial a_j}. \tag{2}$$

Here $\boldsymbol{x}(\boldsymbol{a},t)$ is the position at time t of a particle, which started at $\boldsymbol{a}$ at $t=0$, and ρ is fluid density.

In the kinematic formulation of the problem, to which we will confine our consideration, it is necessary to describe the evolution of magnetic-field statistical characteristics by assuming the motion to be specified. The method of specifying is not significant, but can be, for example, as Lagrangian trajectories $\boldsymbol{x}(\boldsymbol{a},t)$. Here, the kinetic equation (9) (see below) will be used.

The involvement of the density ρ in (2) creates additional difficulties. Methodologically, it is convenient to change from the Lagrangian variables $\boldsymbol{a}$ and t in (2) to the Euler variables $\boldsymbol{x}$ and t. In this case, the initial position $\boldsymbol{a}(\boldsymbol{x},t)$ of a fluid particle will be regarded as a function of the Euler variables:

$$H_i(\boldsymbol{x},t) = \tfrac{1}{2}\epsilon_{ipf}\,\epsilon_{jkm} \frac{\partial a_k}{\partial x_p} \frac{\partial a_m}{\partial x_f} H_j[\boldsymbol{a}(\boldsymbol{x},t),0]. \tag{3}$$

Here ϵ_{ipf} is the antisymmetric tensor and the density of the medium no longer appears. Further simplifications are achieved by introducing a vector potential:

$$A_i(\boldsymbol{x},t) = A_j[\boldsymbol{a}(\boldsymbol{x},t),0] \frac{\partial a_j(\boldsymbol{x},t)}{\partial x_i}, \quad \boldsymbol{H} = \boldsymbol{\nabla}\times\boldsymbol{A}. \tag{4}$$

We shall be interested in the field characteristics that are averaged over an ensemble of random motions. Averaging becomes relatively simple if we omit from (4) the dependence of the vector potential on the random argument $\boldsymbol{a}(\boldsymbol{x},t)$ by introducing convolutions with the δ-function:

$$A_i(\boldsymbol{x},t) = \lim_{{}^1\boldsymbol{x}\to\boldsymbol{x}} {}^1\partial_i \int \mathrm{d}\boldsymbol{z}\, \mathrm{d}\,{}^1\boldsymbol{z}\; {}^1z_j\, \delta[{}^1\boldsymbol{z}-\boldsymbol{a}({}^1\boldsymbol{x},t)]\, \delta[\boldsymbol{z}-\boldsymbol{a}(\boldsymbol{x},t)]\, A_j(\boldsymbol{z},0), \tag{5}$$

$${}^1\partial_i = \frac{\partial}{\partial\,{}^1x_i}.$$

We assume the limit procedure in (5) to exist. This assumption seems natural as (5) is in fact a modified (4). On averaging (5), we obtain the following formula for a large-scale vector potential:

$$\langle A_i(\boldsymbol{x},t)\rangle = \lim_{{}^1\boldsymbol{x}\to\boldsymbol{x}} {}^1\partial_i \int \mathrm{d}\boldsymbol{z}\, \mathrm{d}\,{}^1\boldsymbol{z}\, p_2(\boldsymbol{x},{}^1\boldsymbol{x}\,|\,\boldsymbol{z},{}^1\boldsymbol{z},t)\, \langle A_j(\boldsymbol{z},0)\rangle\, {}^1z_j. \tag{6}$$

In (6) we have assumed the statistical independence of the initial field of random motions. This means that (6) is to be considered asymptotically, at $t \gg \tau$. The quantity p_2 appearing in (6), represents a two-particle-transition probability density, i.e. the density of the probability that at the initial time $t=0$ the particles were at positions $\boldsymbol{z}$ and ${}^1\boldsymbol{z}$ provided that at the present moment t they are at $\boldsymbol{x}$ and ${}^1\boldsymbol{x}$ respectively,

$$p_2(\boldsymbol{x},{}^1\boldsymbol{x}\,|\,\boldsymbol{z},{}^1\boldsymbol{z},t) = \langle \delta[\boldsymbol{z}-\boldsymbol{a}(\boldsymbol{x},t)]\, \delta[{}^1\boldsymbol{z}-\boldsymbol{a}({}^1\boldsymbol{x},t)]\rangle.$$

Thus, the problem of magnetic-field dynamics is now that of determining the statistical properties of fluid particles in a turbulent flow.

Formally, it is possible to state that the relation (6), by itself, is already a solution to the problem. Since the p_2 function depends only on the properties of the turbulence and is independent of the magnetic field, information about the motion of the medium can be specified by choosing some particular function p_2. Such a method was used by Vainshtein (1981) but it is rather complicated. First, it is rather difficult to specify p_2 such as to satisfy all the properties of a joint density of the transition probability of two fluid particles (going to zero when $\boldsymbol{z} = {}^1\boldsymbol{z}$ and $\boldsymbol{x} \neq {}^1\boldsymbol{x}$, a symmetry about the transformation $\boldsymbol{z} \leftrightarrow {}^1\boldsymbol{z}$, $\boldsymbol{x} \leftrightarrow {}^1\boldsymbol{x}$, etc.). Secondly, when a particular function p_2 is specified, a specific model of a turbulent flow is considered but this entails loss of the possibility of describing the field dynamics in a general form. Also, to describe SSF dynamics requires a four-point function p_4 that is still more difficult to specify than p_2. Equation (6) affords only a formal solution to the problem. An equation for p_2 will be derived below and the SSF dynamics will be defined using this equation.

Let us consider a small-scale magnetic field $\boldsymbol{h}$. Averaging separates the large-scale from the small-scale components of the fields:

$$\langle \boldsymbol{v} \rangle = \boldsymbol{V}, \quad \boldsymbol{v} = \boldsymbol{V} + \boldsymbol{u},$$
$$\langle \boldsymbol{H} \rangle = \boldsymbol{B}, \quad \boldsymbol{H} = \boldsymbol{B} + \boldsymbol{h},$$
$$\langle \boldsymbol{A} \rangle = \boldsymbol{A}_0, \quad \boldsymbol{A} = \boldsymbol{A}_0 + \boldsymbol{A}_1.$$

In order to describe the field $\boldsymbol{h}$ we will employ the correlation tensor

$$B_{ij}({}^1\boldsymbol{x}, {}^2\boldsymbol{x}, t) = \langle h_i({}^1\boldsymbol{x}, t)\, h_j({}^2\boldsymbol{x}, t) \rangle. \tag{7}$$

Let us multiply $A_i({}^1\boldsymbol{x}, t)$ by $A_j({}^2\boldsymbol{x}, t)$, express, with the aid of (5), the product in terms of the initial field and average the result. We thus obtain

$$\langle A_i({}^1\boldsymbol{x}, t)\, A_j({}^2\boldsymbol{x}, t) \rangle = \lim_{\substack{{}^3\boldsymbol{x} \to {}^1\boldsymbol{x} \\ {}^4\boldsymbol{x} \to {}^2\boldsymbol{x}}} {}^3\partial_i\, {}^4\partial_j \int \mathrm{d}\,{}^1\boldsymbol{z} \ldots \mathrm{d}\,{}^4\boldsymbol{z}\; p_4({}^\alpha\boldsymbol{x} \,|\, {}^\mu\boldsymbol{z}, t)\, {}^3z_p\, {}^4z_f \langle A_p({}^1\boldsymbol{z}, 0)\, A_f({}^2\boldsymbol{z}, 0) \rangle. \tag{8}$$

Generally, the mean fields contribute to this expression. The contribution is readily excluded with the aid of (6).

To describe the fluctuation fields requires a four-point function of fluid-particle distribution, p_4:

$$p_4({}^\alpha\boldsymbol{x} \,|\, {}^\mu\boldsymbol{z}, t) = \left\langle \prod_{\mu=1}^{4} \delta[{}^\mu\boldsymbol{z} - \boldsymbol{a}({}^\mu\boldsymbol{x}, t)] \right\rangle.$$

Knowledge of the kinetic equations for p_2 and p_4 makes it possible to define the dynamics of the mean and fluctuation magnetic fields.

3. The kinetic equations for fluid particles

The purpose of this section is to substantiate the generalized KFP equations for p_n

$$\frac{\partial p_n}{\partial t} + V_i({}^\alpha\boldsymbol{x})\, {}^\alpha\partial_i\, p_n = T_{ij}({}^\alpha\boldsymbol{x}, {}^\mu\boldsymbol{x})\, {}^\alpha\partial_i\, {}^\mu\partial_j\, p_n. \tag{9}$$

The Greek indices here run as follows; $1, 2, \ldots, n$. A repetition of these indices means summation. This equation for a two-point distribution function was proposed by Roberts (1961) and Kraichnan (1966). It is worth remembering that p_n represents the probability densities with respect to the initial coordinates of particles. The normalization conditions have the form

$$\int p_1(\boldsymbol{x}\,|\,\boldsymbol{a},t)\,\mathrm{d}\boldsymbol{a} = 1,$$

$$\int p_2({}^1\boldsymbol{x},{}^2\boldsymbol{x}\,|{}^1\boldsymbol{a},{}^2\boldsymbol{a},t)\,\mathrm{d}\,{}^2\boldsymbol{a} = p_1({}^1\boldsymbol{x}\,|{}^1\boldsymbol{a},t),$$

and so on and thus the differentiation operators on the right-hand side of (9) stand to the right of the coefficients T_{ij}.

When the motion of fluid particles represents a random Markovian process ($\tau \ll l/u$), (9) is the KFP equation (Kolmogorov 1931). Also,

$$T_{ij}(r) = \int_0^\infty \langle u_i(\boldsymbol{x},t)\,u_j(\boldsymbol{x}-\boldsymbol{r},t-s)\rangle\,\mathrm{d}s. \tag{10}$$

For real turbulence ($\tau \sim l/u$) (9), as we will see, remains valid but (10), generally, becomes invalid.

The most straightforward investigation of generalized KFP equations was made by Pawula (1967). In his paper, the following main statements were proved.

(i) The kinetic equations can be expressed containing time derivatives not higher than first order:

$$\frac{\partial p}{\partial t} = \hat{\mathrm{L}}p, \tag{11}$$

where $\hat{\mathrm{L}}$ is an operator which involves neither time derivatives nor time integrals. A justification for this statement is given in Appendix A.

(ii) There are only two possibilities for $\hat{\mathrm{L}}$ if p in (11) is p_1: $\hat{\mathrm{L}}$ may be a differential operator not higher than second order or an integral operator (a differential operator of infinite order).

(iii) If the kinetic equation for a one-point distribution function is a second-order differential equation, then the kinetic equations for $p_2, p_3, p_4, \ldots$ are also second-order differential equations.

We assume the distribution function p_1 to be normal. Let us consider this assumption in more detail for it is important to the theory developed below. It has not yet been rigorously proved, but seems physically realistic. In fact, p_1 is the distribution function of one-particle displacements. In an asymptotic regime i.e. at $t \gg \tau$ particle displacement may be represented as the sum of a large number of independent displacements, and the distribution tends to a normal one according to law of large number.

This leads to the following kinetic equation:

$$\frac{\partial p_1}{\partial t} = a_i(t)\,\partial_i p_1 + b_{ij}(t)\,\partial_i\,\partial_j\,p_1. \tag{12}$$

According to (iii) above, the kinetic equation for p_n has to be of a generalized KFP type (9). Vector V_i corresponds to mean velocity and coefficients T_{ij} possess correlation tensor properties (see Pawula 1967).

According to (ii) and (iii), there is, in principle, the other possibility that p_2 satisfies an integral kinetic equation. It may be expected that the equation describes a discontinuous stochastic process. It is this situation that corresponds to Markovian processes: continuous random processes are governed by KFP equations and discontinuous ones by integral equations (Rytov 1976).

Some points outlined in support of these being generalized KFP equations that apply to real turbulence are outlined in Appendix B. This may be considered as (not rigorous) justification of the assumption that a one-point distribution function is normal.

Note finally that assuming the statistics of the turbulence to be a one-particle distribution is probably no less generally valid than previous assumptions. In particular, Kraichnan (1976) assumed a normal distribution of Fourier amplitudes of Eulerian velocities.

The substantiation of (9) does not lead to an explicit form of the expression for the tensor T_{ij}. It has only been established that this tensor has correlation properties. But even this, together with (9), allows us to draw important conclusions about the properties of magnetic fields. Also, the tensor T_{ij} can, in principle, be determined experimentally by measuring the correlation characteristics of displacements of two fluid particles in a turbulent flow.

4. The mean field dynamics

An averaged solution of the induction equation (6) together with the kinetic equation for p_2 (9) define the dynamics of large-scale magnetic fields. On differentiating (6) with respect to time, taking into account (9) and using the relation

$$\lim_{{}^1x\to x} ({}^1\partial_i+\partial_i)f = \partial_i \lim_{{}^1x\to x} f \tag{13}$$

we obtain an equation for the vector potential (Kichatinov 1985)

$$\frac{\partial A_{0i}}{\partial t}+V_j\partial_j A_{0i}+A_{0j}\partial_i V_j = D_{\mathrm{T}}\nabla^2\cdot A_{0i}+\alpha\epsilon_{ijk}\partial_j A_{0k}. \tag{14}$$

The coefficients of turbulent diffusion of the magnetic field D_{T} and generation α are expressed in terms of the tensor T_{ij}:

$$D_{\mathrm{T}} = \tfrac{1}{3}T_{ij}(0), \quad \alpha = 2C(0) \tag{15}$$

It is taken into account here that in the present case of isotropic turbulence the tensor T_{ij} has the following structure (Monin & Yaglom 1975):

$$T_{ij}(\boldsymbol{r}) = T_{NN}(r)\left(\delta_{ij}-\frac{r_i r_j}{r^2}\right)+\frac{T_{LL}(r)\, r_1 r_j}{r^2}+C(r)\,\epsilon_{ijf}\, r_f. \tag{16}$$

Applying the operator $\nabla\times$ to (14) we find an equation for the mean magnetic field:

$$\frac{\partial \boldsymbol{B}}{\partial t}-\nabla\times(\boldsymbol{V}\times\boldsymbol{B}) = D_{\mathrm{T}}\nabla^2\cdot\boldsymbol{B}+\nabla\times\alpha\boldsymbol{B}. \tag{17}$$

The left-hand side of (17) involves the term $\nabla\times(\boldsymbol{V}\times\boldsymbol{B})$ which obviously arises in the presence of a large-scale velocity. The first term on the right-hand side describes the eddy diffusion of the magnetic field while the second one, the generation (the so called α-effect (Steenbeck *et al.* 1966)).

As mentioned in the previous section, the tensor T_{ij} possesses correlation properties. For this reason the eddy diffusivity D_{T} of a large-scale magnetic field is positive. This statement follows directly from (15). Note that D_{T} of (15) coincides with the diffusion coefficient of the entropy field (see Appendix A). Parker (1971) has advanced an explanation for such a coincidence. In the case of the Markovian process ($\tau \ll l/u$) this was proved rigorously (Vainshtein 1972). It is evident that the coefficient D_{T} (15) does not include any helicity contribution in explicit form (it may depend on helicity implicitly: the present theory does not predict an evident form of the correlation tensor T_{ij}).

The kinetic equations (9) should be regarded as accurate (though with unknown coefficients). Precisely in the same sense, the formula for D_{T} (15) is accurate, although our only knowledge of the tensor T_{ij} is that it possesses properties of a correlation one. This allows us to look somewhat differently at other representations of D_{T}. According to Appendix A, perturbation theory results in an approximate equation of the type (A6), which contains time integrals. This equation is hardly comparable with (17). First, (A6) includes space derivatives of higher than second order. Moreover, the coefficient D_{T} is expressed as an infinite series in powers of $\tau u/l$. In the zeroth-order approximation ($\tau u/l \to 0$) D_{T} does not depend on helicity but does so in explicit form in the first approximation. But we do not know the whole sum of the series representing D_{T}. Let us consider another approach to the problem. $H_j(\boldsymbol{a}, 0)$ in (2) (in the case of incompressible fluid) can be expressed as

$$H_j(\boldsymbol{a},0) = H_j(\boldsymbol{x},t) - \xi_f \frac{\partial H_i}{\partial x_f} + \tfrac{1}{2}\xi_f \xi_m \frac{\partial^2 H_j}{\partial x_f \partial x_m} + \dots \tag{18}$$

resulting in the following expressions for α and D_{T}:

$$\alpha = \frac{\mathrm{d}\gamma}{\mathrm{d}t}, \quad D_{\mathrm{T}} = \frac{\mathrm{d}}{\mathrm{d}t}(\zeta + \tfrac{1}{2}\gamma); \tag{19}$$

$$\gamma = \left\langle \xi_2 \frac{\partial \xi_1}{\partial a_3} \right\rangle, \quad \zeta = \tfrac{1}{2}\langle \xi_2^2 \rangle \left\langle \xi_2^2 \frac{\partial \xi_1}{\partial a_1} \right\rangle \tag{20}$$

(Moffatt 1974; Kraichnan 1976*b*). When representing α and D_{T} by (15), we consider essentially the asymptotic regime $t \gg \tau$, so that these coefficients are time independent. This presents obstacles to direct comparison of expressions (15) and (19). If $\alpha \to$ const at $t \gg \tau$, then $\gamma \sim t$ and $\frac{1}{2}\mathrm{d}\gamma^2/\mathrm{d}t \sim t$, so that the second term in (19) for D_{T} involves a time dependence. This corresponds to a divergence of D_{T} at $t \to \infty$, as suggested by Moffatt (1974). In fact, the divergence results from expansion (18) and suggests an equation for $\boldsymbol{B}$ of the type (A6), containing a time dependence in explicit form.

The coefficients α and D_{T} for moderate time $t \leqslant 4\tau$ have been obtained from computer simulations by Kraichnan (1976*a*, *b*). Asymptotic results for α and D_{T} from (15) obtained for $t \gg \tau$ are apparently not compared with these simulations.

5. The small-scale-field dynamics

The problem of SSF dynamics in a turbulent conducting fluid was formulated by Batchelor (1950). In order to describe an SSF we need to seek an equation for the magnetic-field correlation tensor B_{ij} and so we shall use (8). We shall have to use the equation for a joint transition probability four-point density (9) ($n = 4$). On differentiating (8) with respect to time and using (9) we obtain an equation for the correlation tensor of a vector potential A_{ij}, and then, with the aid of the formula

$$B_{ij}({}^1\boldsymbol{x}, {}^2\boldsymbol{x}, t) = \epsilon_{ilm}\,\epsilon_{jkn}\,{}^1\partial_l\,{}^2\partial_k\,A_{mn}({}^1\boldsymbol{x}, {}^2\boldsymbol{x}, t)$$

determine an equation for the tensor B_{ij}. In this case it is also necessary to use (14) in order to exclude the contribution of large-scale fields. The final result, omitting cumbersome intermediate calculations, is

$$\left.\begin{aligned}\frac{\partial}{\partial t}B_{ij}({}^1\boldsymbol{x},{}^2\boldsymbol{x},t) &= {}^1\hat{R}_{ipf}\,V_p({}^1\boldsymbol{x})B_{fj}({}^1\boldsymbol{x},{}^2\boldsymbol{x},t)+{}^2\hat{R}_{jpf}\,V_p({}^2\boldsymbol{x})\,B_{if}({}^1\boldsymbol{x},{}^2\boldsymbol{x},t)\\ &\quad+2\,{}^1\hat{R}_{ipm}\,{}^2\hat{R}_{jkn}\,T_{pk}({}^1\boldsymbol{x}-{}^2\boldsymbol{x})\,B_m({}^1\boldsymbol{x})\,B_n({}^2\boldsymbol{x})\\ &\quad+\sum_{\alpha,\mu=1}^{2}{}^\alpha\hat{R}_{imn}\,{}^\mu\hat{R}_{jpf}\,T_{pm}({}^\mu\boldsymbol{x}-{}^\alpha\boldsymbol{x})\,B_{fn}({}^1\boldsymbol{x},{}^2\boldsymbol{x},t),\\ {}^\mu\hat{R}_{ipf} &= \epsilon_{ikl}\,\epsilon_{lpf}\,{}^\mu\partial_k.\end{aligned}\right\} \tag{21}$$

Before proceeding with further transformations we shall discuss the role played by the microscopic magnetic viscosity in the SSF dynamics. For the case of large magnetic Reynolds number, $R_{\mathrm{m}} = (\langle u^2\rangle^{\frac{1}{2}} l/D_0$ (D_0 is the coefficient of microscopic magnetic diffusivity), this diffusivity is unimportant. Let us examine this in greater detail. It is known that the turbulence represents a multi-scale phenomenon: it involves eddies, from the largest with a characteristic scale l, to the smallest having a scale of viscous damping l_χ. The corresponding Reynolds number for the length l_χ is of order unity: $u_\chi l_\chi/\chi \sim$ (χ is viscosity). For larger scales $R > 1$. For the magnetic field, there also exists a region of scales in which $R_{\mathrm{m}} \gg 1$, i.e. the frozen-in condition is fulfilled. If $R_{\mathrm{m}} \gg R$, then the field is frozen in all eddies, down to the smallest of a size l_χ. If, however, $R_{\mathrm{m}} \leqslant R$, then the Ohmic dissipation becomes important, starting from a certain scale $l_{\mathrm{D}}(l_{\mathrm{D}} > l_\chi)$, for which $u_{\mathrm{D}}\, l_{\mathrm{D}}/D_0 \sim 1$. For an arbitrary relation between R_{m} and R, the characteristic-field scale is no less than

$$\max\{l_{\mathrm{D}}, l_\chi\} = l_0.$$

For large R_{m} and R, which is of interest here, there is a range of scales l', $l_0 \ll l' \lesssim l$, in which the field is a frozen-in one. We shall confine our attention solely to such fields. In this sense only the microscopic magnetic diffusion is not essential for the SSF dynamics.

In the range of scales l' (the analogue of an inertial range of hydrodynamic turbulence, for which dissipative effects also play no role) the dynamics of magnetic fields is defined by two competing processes, as mentioned in §1. As will be shown below, the effect of field generation does indeed predominate. Although the microscopic magnetic diffusion plays no role, we will later take into account the finite conductivity of the medium because this is of interest methodologically within the framework of the Lagrangian approach. The finite conductivity is taken into account by introducing a non-helical microturbulence with an external scale l_{m}, $l_{\mathrm{m}} \ll l_0$. With regard to such a microturbulence, the SSFs are considered to be large-scale and therefore, because there is no helicity present, only eddy magnetic diffusion will be effective. In other words, the turbulent medium considered, with a finite conductivity, is replaced with another in which conductivity is infinite, but in addition to the turbulence under consideration there is also microturbulence present. For SSFs both these media are equivalent. In accordance with (15), let us assume that for the microturbulence $T_{ii}(0) = 3D_0$. So, the finite conductivity will be taken into account if the tensor $T_{ij}(\boldsymbol{r})$ involved in (21) is replaced with $T'_{ij}(\boldsymbol{r})$:

$$T'_{ij}(\boldsymbol{r}) = \left\{\begin{array}{ll} T_{ij}(\boldsymbol{r}) & \text{at } r > 0\\ T_{ij}(0)+D_0\,\delta_{ij} & \text{at } r = 0.\end{array}\right\} \tag{22}$$

The structure of (21) is rather complex and in order to analyse its physical content we need to make a number of simplifying assumptions and perform additional transformations. We shall assume the large-scale components of the magnetic field to be zero. In this case, only the last term on the right-hand side of (21) is retained. We shall further assume that the initial distribution of the magnetic field is homogeneous and isotropic. Since the turbulence under consideration also possesses this property, a homogeneity and isotropy of magnetic fluctuations will occur at subsequent moments of time and the tensor B_{ij} takes the form

$$B_{ij}(\boldsymbol{r}) = B_{LL}(r)\frac{r_i r_j}{r^2} + B_{NN}(r)\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right) + \tilde{B}(r)\,\epsilon_{ijf} r_f, \tag{23}$$

$$B_{NN}(r) = \frac{1}{2r}\frac{\partial}{\partial r} r^2 B_{LL}(r), \quad \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2. \tag{24}$$

Here the SSF is taken to be solenoidal, and, therefore, the longitudinal B_{LL} and transverse B_{NN} correlation functions are related by (24).

Let us write (21) taking into account (22) and the assumptions made previously:

$$\frac{\partial}{\partial t} B_{ij}(\boldsymbol{r}, t) = \sum_{\alpha,\mu=1}^{2} {}^{\alpha}\hat{R}_{imn}\, {}^{\mu}\hat{R}_{jpf} T'_{pm}({}^{\mu}\boldsymbol{x} - {}^{\alpha}\boldsymbol{x})\, B_{fn}(\boldsymbol{r}, t). \tag{25}$$

Multiplying (25) by $r_i r_j / r^2$ and by $\epsilon_{ijf} r_f$, taking into account the relations written above and the obvious equalities

$$B_{ij}\frac{r_i r_j}{r^2} = B_{LL}(r), \quad B_{ij}\epsilon_{ijf} r_f = 2r^2 \tilde{B}(r),$$

we obtain a system of two equations (Kichatinov 1985):

$$\frac{\partial B_{LL}}{\partial t} = \frac{2}{r^4}\frac{\partial}{\partial r} r^4 \kappa \frac{\partial B_{LL}}{\partial r} + Q B_{LL} + 4\alpha \tilde{B}; \tag{26}$$

$$\frac{\partial \tilde{B}}{\partial t} = \frac{1}{r^4}\frac{\partial}{\partial r} r^4 \frac{\partial}{\partial r}(2\kappa \tilde{B} - \alpha B_{LL}). \tag{27}$$

Here we have used the notation

$$\left.\begin{aligned} \kappa(r) &= D_0 + \kappa_{\mathrm{T}}(r), \\ \kappa_{\mathrm{T}}(r) &= T_{LL}(0) - T_{LL}(r), \\ \alpha(r) &= 2C(0) - 2C(r), \\ Q &= -4\left(\frac{\mathrm{d}}{\mathrm{d}r}\frac{T_{NN}}{r} + \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r} r T_{LL}\right). \end{aligned}\right\} \tag{28}$$

The system (26), (27) generalizes the Kraichnan & Nagarajan (1967) equation that is valid for the case $\nabla \cdot \boldsymbol{u} = 0$ on the one hand, and the equation for magnetic fluctuations for an acoustic turbulence ($\nabla \times \boldsymbol{u} = 0$) (Vainshtein 1970), on the other. In addition, this system includes helicity effects. We may further state that when $\alpha = 0$, (26), for an incompressible medium, is a generalization of the Kazantsev (1976) equation, derived for the Markovian model, for the case of real non-Markovian turbulence, just as (9) is a generalization of the KFP equations.

We shall now analyse the system (26), (27). The first term on the right-hand side represents turbulent, as well as Ohmic, dissipation. For real turbulence the quantity $\kappa_{\mathrm{T}}(r)$ in (28) decreases with decreasing r. Physically this is because the turbulent viscosity $\kappa_T(r)$ includes eddies, of scale less than r. The smaller r is, the fewer such eddies remain and the viscosity, accordingly, diminishes. Over the scales $r \gg l_0$, the Ohmic dissipation is negligible ($D_0 \ll \kappa_T$) and the frozen-in condition is satisfied. Therefore, a decrease of the SSF energy through the first term on the right-hand side of (26) occurs only in this scale, the energy is transferred into smaller fluctuations and the total energy is conserved. This is one of the two competing processes mentioned above. The second process of field amplification is described by the second term on the right-hand side of (26). This can be confirmed by writing (26) in the limit $r \to 0$:

$$\left.\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} B_{LL}(0) = Q(0)\, B_{LL} + 10 D_0\, B''_{LL}(0), \\ &Q(0) > 0, \quad B''_{LL}(0) = \lim_{r\to 0} \frac{\partial^2}{\partial r^2} B_{LL}(r). \end{aligned}\right\} \tag{29}$$

At the initial moment of time, the correlation length of the magnetic fluctuations is of order l, and the Ohmic term involved in (29) is negligible and hence the original energy grows exponentially. The field scale subsequently decreases, however, which makes the two terms on the right-hand side of the same order of magnitude. Therefore, this equation cannot be used to find any dynamo.

The last component on the right-hand side of (26) corresponds to the known α-effect of a turbulent dynamo (Steenbeck *et al.* 1966; Steenbeck & Krause 1966). For the SSF, the α-effect is not as important as for the dynamics of LSF. The helicity of turbulent motion influences the magnetic-fluctuation-energy distribution in scale but does not lead to the SSF generation. This assertion becomes evident when we look at (29), which lacks any helicity contribution.

To solve the problem of the SSF generation, let us consider the non-helical turbulence, for which $\alpha = 0$ and $\tilde{B} = 0$. Introducing a new function $B = r^2 B_{LL}$ into (26) we obtain an equation with a self-conjugate operator:

$$\frac{\partial B}{\partial t} = \frac{2}{r^2}\frac{\partial}{\partial r} r^4 \kappa \frac{\partial}{\partial r}\frac{B}{r^2} + QB. \tag{30}$$

We shall solve the problem for eigenvalues

$$B(r,t) = \psi(r) \exp(-Et) \tag{31}$$

with the aid of the variational principle. To accomplish this, we need to find a minimum of the functional

$$E = \left\{ 2\int_0^\infty \left[\kappa \left(\frac{\mathrm{d}\psi}{\mathrm{d}r}\right)^2 + \frac{2\kappa_1}{r}\frac{\mathrm{d}\psi^2}{\mathrm{d}r} \right] \mathrm{d}r \right\} \left(\int_0^\infty \psi^2\, \mathrm{d}r \right)^{-1}, \tag{32}$$

$$\kappa_1 = T_{NN}(0) - T_{NN}(r).$$

In the inertial range, where the viscosity is not important, we shall represent the correlation functions by

$$\left.\begin{aligned} &T_{LL} = T - A_L r^{-\gamma}, \quad T_{NN} = T - A_N r^{\gamma}, \\ &T, A_L, A_N > 0, \quad 1 < \gamma < 2, \quad l_x \ll r \ll l. \end{aligned}\right\} \tag{33}$$

The correlation functions (33) have the dimensions of the diffusion coefficient ul and hence the exponent γ is associated with the law of diminishing of the characteristic velocity of eddies u_r, as their characteristic scale r decreases: $u_r r \sim r^\gamma, u_r \sim r^{(\gamma-1)}$. In real turbulence, the energy is concentrated at $r \sim l$, i.e. $u_r^2 \sim r^{2(\gamma-1)}$ and

$2(\gamma-1) > 0$, whence follows a lower bound on the value of γ in (33). On the other hand, the 'energy' of fields $\nabla \times \boldsymbol{u}$ or $\nabla \cdot \boldsymbol{u}$ (i.e. the quantities $\langle(\nabla \times \boldsymbol{u})^2\rangle$, $\langle(\nabla \cdot \boldsymbol{u})^2\rangle$) must be concentrated on a length of viscous damping l_χ. In this case only, there occurs a cascaded transfer of the energy into viscous scales and a further energy dissipation there. This implies $u_r^2/r \sim r^{2(\gamma-2)}$ and $2(\gamma-2) < 0$. From this follows an upper bound on the value of γ in (33).

We shall substitute the correlation functions (33) into (16) and require that the correlation functions of the fields $\nabla \times \boldsymbol{u}$ and $\nabla \cdot \boldsymbol{u}$ be positive when $r \to 0$. Then, we shall obtain bounds on the quantities A_L and A_N:

$$\frac{1}{\gamma-1} \leqslant \frac{A_N}{A_L} \leqslant \frac{\gamma+2}{2}. \tag{34}$$

The equality on the upper limit (34) corresponds to an incompressible flow ($\nabla \cdot \boldsymbol{u} = 0$). The equality on the lower limit (34) corresponds to a potential flow ($\nabla \times \boldsymbol{u} = 0$).

From the self-conjugate form of the right-hand side of (30) follows the reality of eigenvalues of E_n. If one of $E_n < 0$, then, according to (31), there is an exponential growth of the SSF, i.e. a dynamo takes place. In order to demonstrate that the dynamo does indeed take place ($E_0 < 0$), it is sufficient to find a trial function that makes the functional (32) negative. The trial function may be an arbitrary one but we shall seek it in such a form that the Ohmic dissipation makes a negligible contribution to the functional (32). Since the Ohmic dissipation acts at very small r, the trial function must tend to zero when $r \to 0$. With such a trial function, the Ohmic dissipation in (32) is not significant and can be neglected. Besides, it is convenient to use a representation of the correlation functions T_{LL} and T_{NN} in (32) in terms of (33) and, therefore, the trial function must also rather rapidly diminish when $r \to l$.

We shall choose ψ in the form

$$\psi = \left(\frac{L}{r}\right)^{\beta} \exp\left(-\frac{L}{r}\right) \quad (l_0 \ll L \ll l). \tag{35}$$

With the aid of (33) we see that the functional (32) becomes negative when

$$\frac{\gamma-1}{2} < \beta < (\gamma-1)\left[4\frac{A_N}{A_L}+1-\frac{\gamma}{2}\right]. \tag{36}$$

Using (33) and (34) we see that the expression between the square brackets in (36) is positive. Therefore, the interval of positive values of β (that satisfy the requirement of smallness of ψ when $r \sim l$) does indeed exist.

Note that when the Ohmic dissipation is taken into account, the functional (32) involves a positive term

$$D_0 \int_0^\infty \left(\frac{\mathrm{d}\psi}{\mathrm{d}r}\right)^2 \mathrm{d}r \Big/ \int_0^\infty \psi^2 \,\mathrm{d}r,$$

that is small as compared with those taken into account when $R_\mathrm{m} \gg 1$.

Thus, the conclusion about the existence of a turbulent dynamo of the SSF at rather large magnetic Reynolds numbers is valid not only for an incompressible fluid and acoustic turbulence: it is generalized to an arbitrary compressible motion. The determination of a critical value of R_m, which is a dynamo-instability threshold, is beyond the scope of this paper.

6. Conclusion

As we have seen, a treatment of the magnetic-field dynamics of a highly conducting turbulent fluid in Lagrangian variables involves kinetic equations for fluid-particle joint transition probability densities.

There are well-known kinetic equations governing Markovian processes – these are the KFP equations of the diffusion type. A number of authors (Bartlett 1955; Pawula 1967; Rytov 1976) have arrived at the conclusion that for processes with a probability consequence (non-Markovian processes), equations are valid that coincide in their form with KFP ones, but having different coefficients; these are generalized KFP equations. This question has been examined in greatest detail by Pawula (1967). The kinetic equations for joint probability densities of transition of n fluid particles have been written in §3. The kinetic equations and correlation properties of their coefficients suggest a number of conclusions which are important for the dynamo theory of magnetic fields.

In the framework of this theory eddy diffusivity of a mean magnetic field is positive and coincides with that of an entropy field (a special case of scalar field defined in Appendix A). The coefficients in general depend on helicity in implicit form. For large magnetic Reynolds numbers, the dynamo effect of small-scale fields in the inertia range of scales takes place for an arbitrary turbulence.

The authors are grateful to Professors A. M. Yaglom, V. I. Tatarsky and E. N. Parker for helpful discussions, and to Dr R. H. Kraichnan for valuable comments.

Appendix A

It is worthwhile to point to an association of the distribution function p_n with correlation properties of a scalar field that is quite useful here. Let us consider the quantity $S(\boldsymbol{x}, t)$ that satisfies the equation

$$\frac{\partial S}{\partial t} + v_i \partial_i S = 0. \tag{A 1}$$

We will call the quantity S the entropy field because, for an adiabatic flow, equation (A 1) is satisfied by the entropy density. For an incompressible flow, the entropy field possesses the same properties as does the usual scalar field $\theta(\boldsymbol{x}, t)$ satisfying the equation

$$\frac{\partial \theta}{\partial t} + \partial_i v_i \theta = 0.$$

The product of n values of S, taken at n different points, obeys the equation

$$\left.\begin{aligned} \frac{\partial S'_n}{\partial t} + v_i({}^{\alpha}\boldsymbol{x}, t)\, {}^{\alpha}\partial_i S'_n &= 0, \\ S'_n &= \prod_{\alpha=1}^{n} S({}^{\alpha}\boldsymbol{x}, t). \end{aligned}\right\} \tag{A 2}$$

It is easy to see that the equation for a 'non-averaged' probability density of transition of fluid particles p'_n coincides in structure with (A 2):

$$\left.\begin{aligned} &\frac{\partial p'_n}{\partial t}+v_i({}^\alpha\boldsymbol{x},t)\,{}^\alpha\partial_i\,p'_n=0,\\ &p'_n({}^\alpha\boldsymbol{x}\,|\,{}^\alpha z,t)=\prod_{\alpha=1}^{n}\delta[{}^\alpha\boldsymbol{z}-\boldsymbol{a}({}^\alpha\boldsymbol{x},t)]\end{aligned}\right\} \tag{A 3}$$

(Liouville theorem). It becomes apparent from equations (A 2) and (A 3) that the kinetic equation for p_n and the equation for the n-point correlation function of the entropy field $S=\langle S'_n\rangle$ should also coincide. Integration of (A 2) over time leads to

$$S'_n(t)=S'_n(t_0)-\int_{t_0}^{t}v_i({}^\alpha\boldsymbol{x},t')\,{}^\alpha\partial_i\,S'_n(t')\,\mathrm{d}t'.$$

Applying the iteration procedure to this equation we can express $S'_n(t)$ through $S'_n(t_0)$:

$$S'_n(t)=\hat{\Sigma}S'_n(t_0)$$

and
$$S_n=\langle S'_n(t)\rangle=\langle\hat{\Sigma}\rangle\langle S'_n(t_0)\rangle=\langle\hat{\Sigma}\rangle\,S_n(t_0), \tag{A 4}$$

where $\hat{\Sigma}$ is an integral operator (or a differential operator of infinite order). It follows from (A 4) that

$$\frac{\partial S_n}{\partial t}=\frac{\partial\langle\hat{\Sigma}\rangle}{\partial t}S_n(t_0),$$

$$S_n(t_0)=(\langle\hat{\Sigma}\rangle)^{-1}S_n(t),$$

where $(\langle\hat{\Sigma}\rangle)^{-1}$ is a reverse operator, so that $\langle\hat{\Sigma}\rangle(\langle\hat{\Sigma}\rangle)^{-1}=1$. Finally we have

$$\left.\begin{aligned} &\frac{\partial S_n(t)}{\partial t}=\hat{L}S_n(t),\\ &\hat{L}=\frac{\partial\langle\hat{\Sigma}\rangle}{\partial t}(\langle\hat{\Sigma}\rangle)^{-1}.\end{aligned}\right\} \tag{A 5}$$

Note that the operator $\hat{L}$ does not involve time derivatives or an integral over past time. We have given above a simplified proof of the possible form of (A 5) of an equation for S_n (and for p_n). In a more general and detailed analysis by Pawula (1967) the distribution function $p(t)$ is expressed in terms of $p(t-\Delta t)$, $\Delta t\to 0$. In this way he arrived at an equation of the type (A 5) with no reverse operator $(\langle\hat{\Sigma}\rangle)^{-1}$. Note that perturbation theory deals with the expansion of operator $\langle\hat{\Sigma}\rangle$ involving no reverse operator. This leads to an approximate equation involving a time integral (see e.g. Vainshtein 1972)

$$\frac{\partial}{\partial t}S_n(t)=\int_0^t\hat{M}(t-\tau)\,S_n(\tau)\,\mathrm{d}\tau. \tag{A 6}$$

Appendix B

Let us discuss the possibility that p_1 is not a Gaussian function and satisfies some integral equation (the differential equation of infinite order). As we have seen, in this case the equations for all $p_n (n \leqslant 4)$ immediately become integral (see Pawula 1967, §3, statement 3). We can demonstrate that the use of such equations leads to physically meaningless results by examining an example of a statistically homogeneous distribution of an entropy field.

The equation for p_2 holds for S_2, in particular, when the dependence of S_2 on ${}^1\boldsymbol{x}$ and ${}^2\boldsymbol{x}$ is in the form $r = |{}^1\boldsymbol{x}-{}^2\boldsymbol{x}|$. The general form of this equation is

$$\begin{aligned}\frac{\partial}{\partial t}S_2(r) &= \int K(r,y)\,S_2(y)\,\mathrm{d}y \\ &= \int \tilde{K}(r-y,y)\,S_2(y)\,\mathrm{d}y \\ &= \left(a_1\frac{\partial}{\partial r}+a_2\frac{\partial^2}{\partial r^2}+a_3\frac{\partial^3}{\partial r^3}+\ldots\right)S_2(r). \qquad \text{(B 1)}\end{aligned}$$

Here a_m is a function of r. The series in (B 1) can be obtained by Taylor expansion of $S_2(y)$ in the vicinity of point r.

The dimensions of the a_m functions are: a_1 coincides with that of u; a_2 with ul; and a_m with $ul^{(m-1)}$. Let us clarify the behaviour of the a_m functions when $r \ll l$. It is clear that $a_m \sim r^m$. The point here is that the time of change of S_2, for very small r, $r < l_\chi (l_\chi$ is the smallest scale in the turbulence spectrum, defined by viscous damping) coincides in order of magnitude with the eddy turnover time l_χ/u_χ (u_χ is the characteristic rate of fluctuations of a scale l_χ). If the expansion of the a_m functions into a power series of r started from powers lower than m, then a change of S_2 for $r \ll l$ would proceed much more rapidly. For example, if at small r $a_2 \sim r$ were valid, i.e. $a_2 = a_2' r$, then the series (B 1) would lead to a change of S_2 for the time $r/u \ll l_\chi/u_\chi$ (because $a_2' \approx u$). Let us write the series (B 1) for small r:

$$\frac{\partial}{\partial t}S_2(r) = T_2\left(b_1 r\frac{\partial}{\partial r}+b_2 r^2\frac{\partial^2}{\partial r^2}+b_3 r^3\frac{\partial^3}{\partial r^3}+\ldots\right)S_2(r) \qquad \text{(B 2)}$$

or in the spectral representation:

$$\frac{\partial}{\partial t}S_2(k) = T_2\left(\tilde{b}_0+\tilde{b}_1 k\frac{\partial}{\partial k}+\tilde{b}_2 k^2\frac{\partial^2}{\partial k^2}+\ldots\right)S_2(k). \qquad \text{(B 3)}$$

Here $T_2 \approx u_\chi/l_\chi$, and the coefficients b_m and $\tilde{b}_m$ are dimensionless constants. Equation (B 2) holds when $r \ll l_\chi$ while (B 3), when $\Delta k l_\chi \gg 1$, Δk being the characteristic scale of variation of $S_2(k)$ at large k, $kl_\chi \gg 1$.

Let us expand the nucleus K of equation (B 1) in a somewhat different way. Taking into account the fact that $S_2(r)$ is an even function, the nucleus K can be symmetrized: it must be even with respect to both the first and second arguments. Let us represent $S_2(y)$ in powers of y and $K(r,y)$, in powers of r:

$$\frac{\partial}{\partial t}S_2(r) = \int (r^2K_2(y)+r^4K_4(y)+\ldots)(S_2^{(0)}+y^2S_2^{(2)}+y^4S_2^{(4)}+\ldots)\,\mathrm{d}y. \qquad \text{(B 4)}$$

Let us add the coefficients at r^{2n} and, using the properties of the a_m functions in (B 1) at small r we have

$$\int K_{2n}(y)\,y^{2p}\,\mathrm{d}y = 0 \qquad \text{(B 5)}$$

for $p > n$, and all terms with $p \leqslant n$ do not go to zero. For the Fourier-transform $\tilde{K}_{2n}(k)$ of the function $K_{2n}(y)$ the property of (B 5) implies that the expansion of $\tilde{K}_{2n}$ in powers of k breaks with the component, that is proportional to k^{2n}. In other words, $K_{2n}(y)$ is not a regular function. It represents the sum of a finite number of components that are proportional to the δ-function and its derivatives up to second-order. This suggests that $K(x,y)$ represents a singular function, consisting of a finite number of δ-functions and its derivatives, i.e. (B 1) is a differential one. As such, as shown by Pawula (1967), it cannot be higher than second-order.

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